52. [*HM11*] Prove that Abel's binomial formula (16) is not always valid when n is not a nonnegative integer, by evaluating the right-hand side when $n = x = -1$, $y = z = 1$.

53. [*M25*] (a) Prove the following identity by induction on m, where m and n are integers:

$$\sum_{k=0}^{m}\binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) = (m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m}.$$

(b) Making use of important relations from exercise 47,

$$\binom{-1/2}{n} = \frac{(-1)^n}{2^{2n}}\binom{2n}{n}, \quad \binom{1/2}{n} = \frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n} = \frac{(-1)^{n-1}}{2^{2n-1}(2n-1)}\binom{2n-1}{n} - \delta_{n0},$$

show that the following formula can be obtained as a special case of the identity in part (a):

$$\sum_{k=0}^{m}\binom{2k-1}{k}\binom{2n-2k}{n-k}\frac{-1}{2k-1} = \frac{n-m}{2n}\binom{2m}{m}\binom{2n-2m}{n-m} + \frac{1}{2}\binom{2n}{n}.$$

(This result is considerably more general than Eq. (26) in the case $r = -1$, $s = 0$, $t = -2$.)

54. [*M21*] Consider Pascal's triangle (as shown in Table 1) as a matrix. What is the *inverse* of that matrix?

55. [*M21*] Considering each of Stirling's triangles (Table 2) as matrices, determine their inverses.

56. [*20*] (*The combinatorial number system.*) For each integer $n = 0, 1, 2, \ldots, 20$, find three integers a, b, c for which $n = \binom{a}{1} + \binom{b}{2} + \binom{c}{3}$ and $0 \le a < b < c$. Can you see how this can be continued for higher values of n?

▸ **57.** [*M22*] Show that the coefficient a_m in Stirling's attempt at generalizing the factorial function, Eq. 1.2.5–(12), is

$$\frac{(-1)^m}{m!}\sum_{k\ge 1}(-1)^k\binom{m-1}{k-1}\ln k.$$

58. [*M23*] In the notation of Eq. (40), prove the "q-nomial theorem":

$$(1+x)(1+qx)\ldots(1+q^{n-1}x) = \sum_k \binom{n}{k}_q q^{k(k-1)/2}x^k.$$

Find q-nomial generalizations of the fundamental identities (17) and (21).

59. [*M25*] A sequence of numbers A_{nk}, $n \ge 0$, $k \ge 0$, satisfies the relations $A_{n0} = 1$, $A_{0k} = \delta_{0k}$, $A_{nk} = A_{(n-1)k} + A_{(n-1)(k-1)} + \binom{n}{k}$ for $nk > 0$. Find A_{nk}.

▸ **60.** [*M23*] We have seen that $\binom{n}{k}$ is the number of combinations of n things, k at a time, namely the number of ways to choose k different things out of a set of n. The *combinations with repetitions* are similar to ordinary combinations, except that we may choose each object any number of times. Thus, the list (1) would be extended to include also *aaa*, *aab*, *aac*, *aad*, *aae*, *abb*, etc., if we were considering combinations with repetition. How many k-combinations of n objects are there, if repetition is allowed?

52. $\sum_{k\ge 0}(k+1)^{-2} = \pi^2/6$. [M. L. J. Hautus observes that the sum is absolutely convergent for all complex x, y, z, n whenever $z \neq 0$, since the terms for large k are always of order $1/k^2$. This convergence is uniform in bounded regions, so we may differentiate the series term by term. If $f(x,y,n)$ is the value of the sum when $z=1$, we find $(\partial/\partial y)f(x,y,n) = nf(x,y,n-1)$ and $(\partial/\partial x)f(x,y,n) = nf(x-1,y+1,n-1)$. These formulas are consistent with $f(x,y,n) = (x+y)^n$; but actually the latter equality seems to hold rarely, if ever, unless the sum is finite. Furthermore the derivative with respect to z is almost always nonzero.]

53. For (b), set $r = \frac{1}{2}$ and $s = -\frac{1}{2}$ in the result of (a).

54. Insert minus signs in a checkerboard pattern as shown.

$$\begin{pmatrix} 1 & -0 & 0 & -0 \\ -1 & 1 & -0 & 0 \\ 1 & -2 & 1 & -0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

This is equivalent to multiplying a_{ij} by $(-1)^{i+j}$. The result is the desired inverse, by Eq. (33).

55. Insert minus signs in one triangle, as in the previous exercise, to get the inverse of the other. (Eq. (47).)

56. 012 013 023 123 014 024 124 034 134 234 015 025 125 035 135 235 045 145 245 345 016. With c fixed, a and b run through the combinations of c things two at a time; with c and b fixed, a runs through the combinations of b things one at a time.

Similarly, we could express all numbers in the form $n = \binom{a}{1} + \binom{b}{2} + \binom{c}{3} + \binom{d}{4}$ with $0 \le a < b < c < d$; the sequence begins 0123 0124 0134 0234 1234 0125 0135 0235 $\ldots$. We can find the combinatorial representation by a "greedy" method, first choosing the largest possible d, then the largest possible c for $n - \binom{d}{4}$, etc. [Section 7.2.1 discusses further properties of this representation.]

58. By induction, since

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q q^{n-k} = \binom{n-1}{k}_q q^k + \binom{n-1}{k-1}_q.$$

It follows that the q-generalization of (21) is

$$\sum_k \binom{r}{k}_q \binom{s}{n-k}_q q^{(r-k)(n-k)} = \sum_k \binom{r}{k}_q \binom{s}{n-k}_q q^{(s-n+k)k} = \binom{r+s}{n}_q.$$

And the identity $1-q^t = -q^t(1-q^{-t})$ makes it easy to generalize (17) to

$$\binom{r}{k}_q = (-1)^k \binom{k-r-1}{k}_q q^{kr-k(k-1)/2}.$$

The q-nomial coefficients arise in many diverse applications; see, for example, Section 5.1.2, and the author's note in *J. Combinatorial Theory* **A10** (1971), 178–180.

Useful facts: When n is a nonnegative integer, $\binom{n}{k}_q$ is a polynomial of degree $k(n-k)$ in q with nonnegative integer coefficients, and it satisfies the reflective laws

$$\binom{n}{k}_q = \binom{n}{n-k}_q = q^{k(n-k)}\binom{n}{k}_{q^{-1}}.$$

If $|q| < 1$ and $|x| < 1$, the q-nomial theorem holds when n is an arbitrary real number, if we replace the left-hand side by $\prod_{k\ge 0}\bigl((1+q^k x)/(1+q^{n+k}x)\bigr)$. Properties of power